Synergetics, Selfsimilarity and Computergraphics

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Selfsimilarity as an intrinsic symmetry of complex dynamical systems is demonstrated using several iteration procedures e. g. time-iteration of discrete maps as well as iterations in space leading to péano curves.

The rapid development of computers during the last decade led to the occurrence of new scientific disciplines, e. g. experimental mathematics. Also in physics, especially in the theory of nonlinear dynamical systems it became possible to obtain solutions numerically which couldn't be calculated using analytical methods like perturbation theories. An important feature of such systems is a property called selfsimilarity, an intrinsic symmetry entirely different from the classical symmetries of translation or rotation. Selfsimilarity means an invariance of certain properties under scale transformations. To get insight into the underlying structures of the complexity occurring in these systems new methods of representation are necessary. It turned out that real-time computergraphics serves as the most efficient tool for the visualization of the spatio-temporal evolution of a nonlinear dynamical system.

An example of a system representing selfsimilar structure in nature is a turbulent flow which contains vortices on each length scale. The question whether it is possible to get a complete description of the whole evolution of such a flow in time and space from the Navier-Stokes equation is not answered, in general, yet. However, during the last years of research about dynamical systems it could be shown that "simply looking" equations can behave in a complex fashion, if they contain non-linear terms.

Let us first discuss the common features of a great class of systems with complex dynamics which we call synergetic systems [1], [2]. These systems consist of a great number of subsystems which are coupled in a nonlinear fashion. The essential point is that these systems are far from thermal equilibrium, i. e. there is a steady flux of e. g. energy, mass or information through the system. This flux can be controlled from outside by external parameters, the so called "control-parameters".

To be as concrete as possible we will look on the Bénard-Problem [3], the onset of thermal convection in a horizontal fluid layer heated from below, which is the most famous example of pattern formation in hydrodynamics. An unstable situation which leads to the convection occurs because the density of the fluid at the bottom of the layer is, due to the higher temperature, smaller than the density at the top and therefore in contrast to a stable arrangement in the layer. What happens, if we change the temperature gradient – the control-parameter – from zero to larger values? If the temperature gradient is sufficiently small, only heat-conduction occurs because the fluid motion is damped by the internal forces of viscosity. But as the temperature gradient exceeds a certain threshold, regular convective motion sets in.

At the critical temperature gradient a qualitative change of the behavior of the system occurs – the symmetry of the solution is spontaneously broken. This is another criterion for a synergetic system i. e. a smooth change of a control-parameter may lead to a qualitativ change of the behavior of the system which usually leads to a reduced symmetry of the solution.

From a theoretical point of view, the problem above is described by the wellknown Navier-Stokes equation and the equation of heat conduction. These are coupled nonlinear partial differential equations, and an exact solution in the whole parameter range would be an unsolvable task, in general. Nevertheless, using linear stability analysis the critical temperature gradient can be calculated. But the behavior of the system beyond this point can not be obtained in that way [4]. Applying the slaving principle of synergetics, the infinitely many
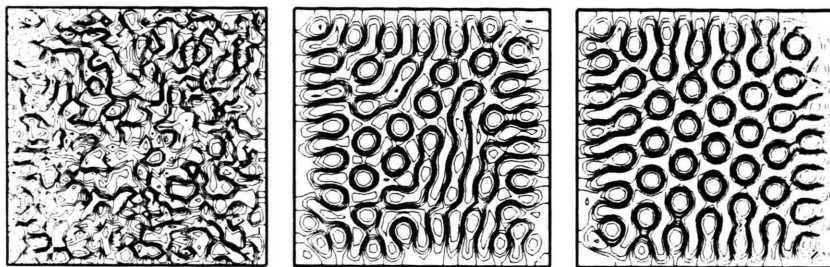
Fig. 1. Bénard-Problem: Evolution in time of the temperature field. See M. Bestehorn and H. Haken. Z. Phys. B **57**, 329 (1984).
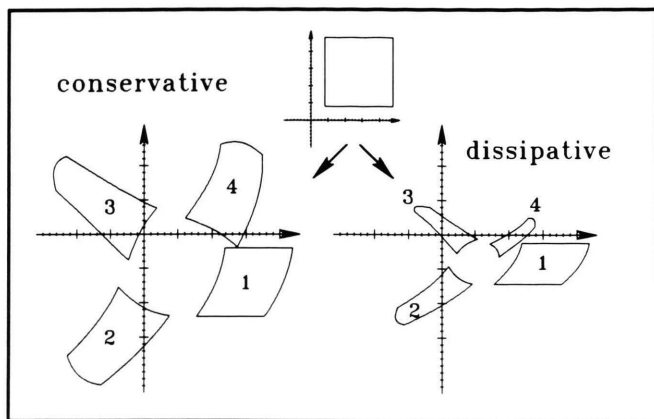

Fig. 2. Hénon map: Behavior of a volume in phase space under iteration. Conservative case (Parameters: $A = 0.4$, $B = -1$). – Dissipative case (Parameters: $A = 0.4$, $B = -0.6$).

degrees of freedom are reduced and the solution close to, but beyond the critical temperature gradient can be derived [5]. In this way it is possible to obtain solutions of threedimensional problems on medium sized computers. Figure 1 shows a typical pattern, obtained by using the methods of synergetics. Starting from an initial state, with a random velocity distribution, the evolution in time leads to the regular hexagonal pattern, which agrees excellently with experimental results. A further increase of the temperature gradient leads to the occurrence of higher instabilities and eventually to the onset of turbulence.

Being aware of the mathematical difficulties which arise when dealing with partial differential equations important features of nonlinear behaviour can already be obtained by studying discrete maps. Because they contain no spatial derivatives and only discrete time intervals, they are easily treatable on computers. The complexity of their behavior can be visualized by graphical representation of the corresponding results. One of the easiest and extensively treated nonlinear map is the logistic map:

$$X_{n+1} = A X_n (1 - X_n)$$

in which selfsimilarity and period doubling was found by Feigenbaum [6], Grossmann and Thomae [7].

Here, we will give an overview of a two-dimensional nonlinear map, the so called Hénon map:

$$X_{n+1} = 1 - A X_n^2 + Y_n,$$
$$Y_{n+1} = B X_n. \tag{1}$$

There are two cases we have to distinguish: For $|B| = 1$ the map is conservative, for $|B| < 1$ dissipative i. e. a volume in phase space contracts under iteration (Figure 2).

The conservative case contains a great variety of interesting phenomena, but since we are interested mainly in symmetries, we will turn our attention to selfsimilarity. Figure 3 shows blowups of a certain region in phase space and it can be seen, that there are selfsimilar structures after each step of enlargement.

To study the dissipative case we fix the parameter B in (1) to a certain value (0.3) and look what happens by varying the parameter A which now acts as a control-parameter. If we increase A from zero and start from the interior of a certain

area in the xy-plane an iteration of the map will lead to a fixpoint. This region is therefore called the basin of attraction of this fixpoint. Starting from any point outside, an iteration diverges.

By increasing A no qualitative change of the behavior is observed, that means the number of attractive points (excluding infinity) is one, until a bifurcation occurs. Now there are two fixpoints and the solution jumps between at each iteration step. A further increase of A leads to another bifurcation with a solution of period four and so on. The

doubling sequence is shown in Fig. 4 and we see that the range of A from one bifurcation to the next gets smaller and smaller and beyond a certain threshold chaos sets in. But, as demonstrated in the same figure, the occurrence of chaos at a certain A does not mean chaos for all greater A. Beyond the first chaotic region a window with periodic solutions appears.

In the xy-plane the solution in the chaotic region of the parameter is a so called strange attractor. It has a fractal dimension between one and two, that
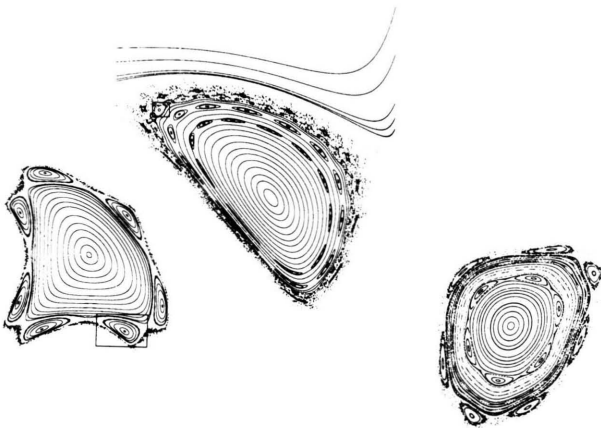
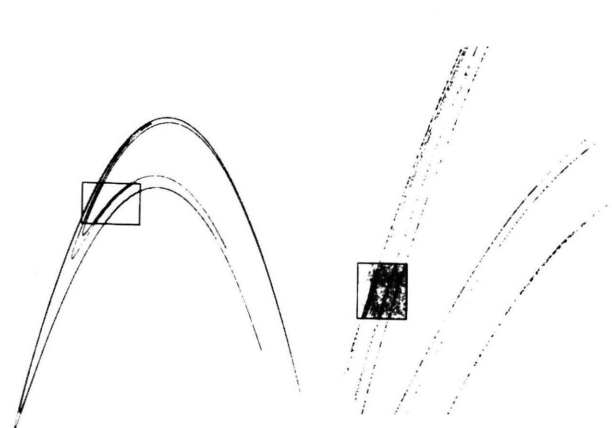Fig. 3. Phase space portraits of the conservative Hénon map.

Fig. 5. The Hénon attractor with blowups showing the self-similar structure (Parameter: $A = 1.4$, $B = 0.3$).
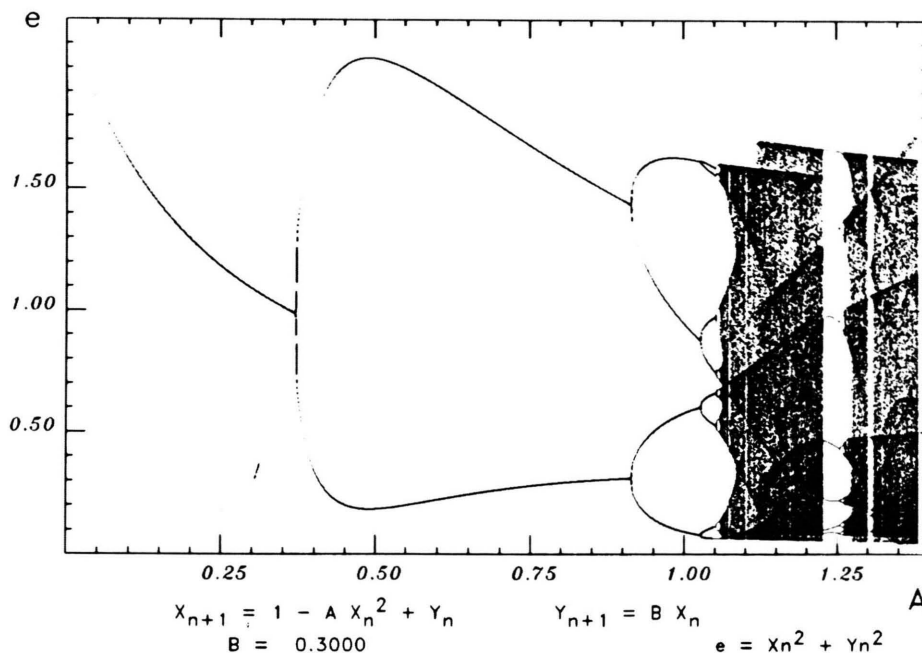
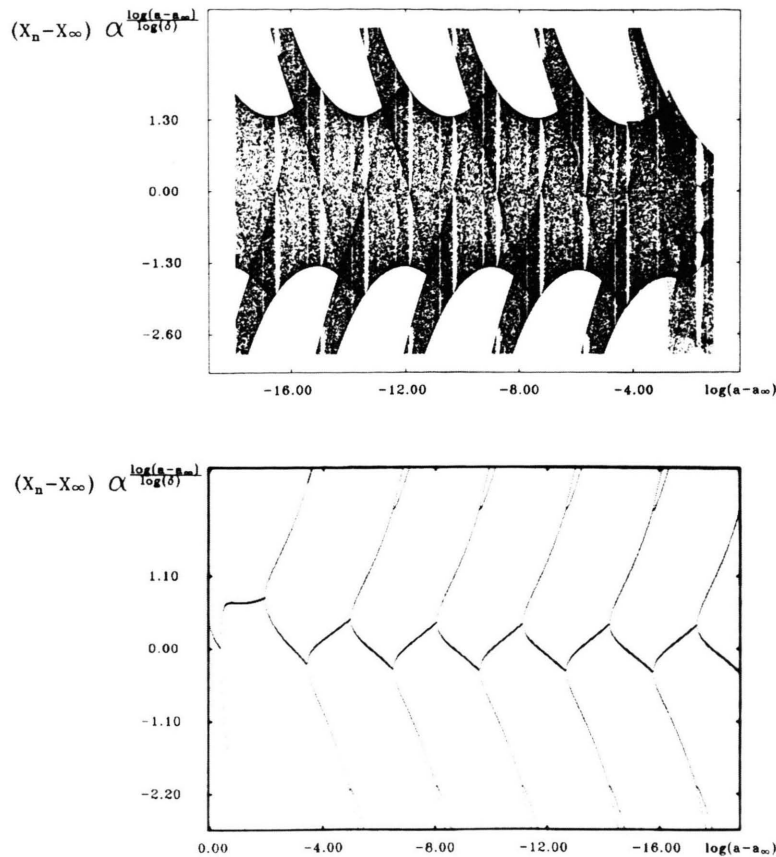Fig. 4. Bifurcation diagram of the Hénon map with $B = 0.3$.

$$X_{n+1} = 1 - A X_n^2 + Y_n \qquad Y_{n+1} = B X_n$$
$$B = 0.3000 \qquad e = X_n^2 + Y_n^2$$

$(X_n - X_\infty) \, \alpha^{\frac{\log(a-a_\infty)}{\log(\delta)}}$

$(X_n - X_\infty) \, \alpha^{\frac{\log(a-a_\infty)}{\log(\delta)}}$

Fig. 6. Bifurcation diagram of the Hénon map with $B = 0.3$. A proper rescaling and a logarithmic axis for A visualizes the selfsimilar behavior in a periodic as well as in a chaotic range for the parameter A. The scaling parameters are: $A_\infty = 1.058049\ldots$, $X_\infty = 0.94601\ldots$, $\delta = 4.669203\ldots$. H. Bunz private communication.
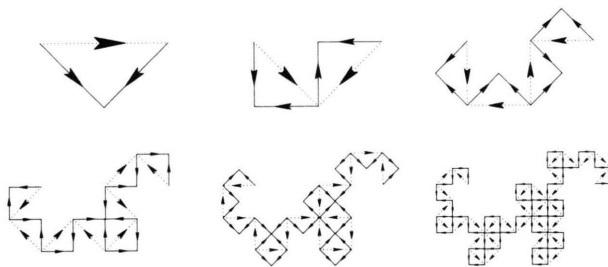

Fig. 7. A generalized Koch curve construction leading to a shape with a fractal border.

Fig. 8. Selfsimilarity of the "dragon".

means this structure is neither a line, nor a plane, but something in between. A plot of such an attractor with blowups of a certain region showing the selfsimilar structure is given in Figure 5.

So far we have looked at selfsimilar structures in the phase space of the Hénon map. However, there exists another selfsimilar behavior in parameter space. In Fig. 4 the quantity $e = X_n^2 + Y_n^2$ is plottet versus the parameter A in a linear scale. If the axis are rescaled appropriately as done in Fig. 6 using a logarithmic scale the selfsimilarity becomes visible in the periodic region as well as in the chaotic one.

Now we turn to the study of iterations in space, which leads us to the paradigm of selfsimilar structures, the world of péano curves, extensively discussed by Mandelbrot [8]. As an example we will look at a special curve, the so called "dragon", one of the simplest structures that can be obtained by using a generalized Koch construction. The iteration procedure is examplified in Figure 7. A line which has a well defined orientation is substituted by two newly orientated and rescaled lines

in a first step. The same substitution is then done with each of the both new lines and so on. After a finite step of iteration the dragon shape gets visible. It can be segmented into portions, which are similar to each other as well as to the whole dragon (Figure 8). If the iteration procedure is done up to infinity, a finite area is densely covered and the length of the boundary line is infinite.

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